Invariance principle:

Example: (pendelum)

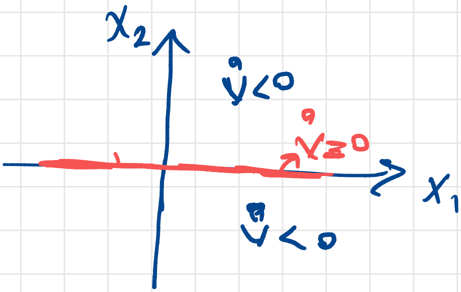
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a\sin(x_1) - bx_2$$

- we tried $V(x) = \frac{1}{2}x_2^2 + a(1-\cos(x_1))$

$$\Longrightarrow \quad \dot{V}(x) = -bx_2^2 \leq 0$$

- $\dot{V}(x) = 0$ for all x on $x_2 = 0$ line

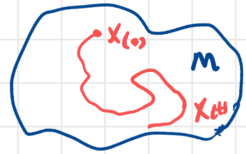
  $\dot{V}(x) < 0$ otherwise



- LaSalle's invariance thm provides a tool to conclude about convergence of asymptotic behavior of solution when $\dot{V} \leq 0$

Definitions:

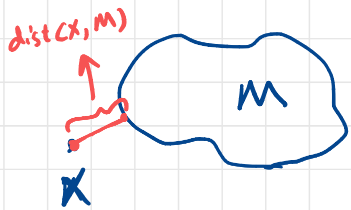
- a set M is invariant under dyn. $\dot{x} = f(x)$ if $x(0) \in M \Rightarrow x(t) \in M \ \forall t$



- $x(t)$ approaches the set M, or $x(t) \to M$ as $t \to \infty$

if $\text{dist}(x(t), M) \to 0$ as $t \to \infty$

$$\text{dist}(x, M) = \inf_{p \in M} \| x - p \|$$



LaSalle theorem (thm 4.4 in Khalil)

- Suppose the solution $x(t)$ is bounded

or $x(t) \in \Omega \ \forall t$ for some compact set Ω

closed and bounded

- Assume $\dot{V}(x) \leq 0 \ \forall x \in \Omega$
- Let $E = \{ x \in \Omega \mid \dot{V}(x) = 0 \}$
- Let M be largest invariant set in E
- Then,

$$x(t) \to M \quad \text{as} \quad t \to \infty$$

Remark

- V is not necessarily p.d. in the theorem.
- Assumption that $X(t)$ is bdd is important.

To ensure boundedness

- If V is p.d., then solutions starting near origin remain bounded (because of stability)
- If V is p.d + radially unbounded, then all solutions remain bounded.

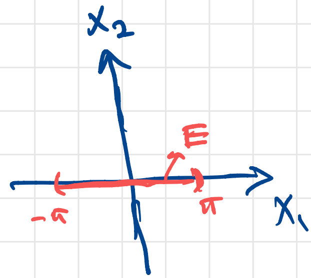
Corollary:

- Let V be p.d and $\dot{V}(x) \le 0 \quad \forall x \in D$
- Let M be largest invariant set of $E = \{x \in D \mid \dot{V}(x) = 0\}$
- If $M = \{0\} \Rightarrow x(t) \to 0$ as $t \to \infty$ $\Rightarrow x = 0$ is AS
- If, moreover, V is radially unbounded and

$$\dot{V}(x) \le 0 \quad \forall x \in \mathbb{R}^n \Rightarrow x = 0 \text{ is GAS}$$

- Remark: $M = \{0\}$ means that no solution can stay in E other than $X(t) = 0$

Example: (pendelum)



$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -a\sin(x_1) - b x_2$$

take $D = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (-\pi, \pi)\}$

$$\dot{V}(x) = -b x_2^2 \leq 0 \quad \forall x \in D$$

$$E = \{x \in D \mid \dot{V}(x) = 0\} = \{(x_1, x_2) \mid x_1 \in (-\pi, \pi),\ x_2 = 0\}$$

- we want to find the largest invariant set M in E

if $x(t) \in E \quad \forall t \Rightarrow x_2(t) = 0 \quad \forall t$

$\Rightarrow \dot{x}_2(t) = 0 \quad \forall t$

$\Rightarrow \sin(x_1(t)) = 0 \quad \forall t$

$\Rightarrow x_1(t) = 0$

$\Rightarrow$ the only solution that stays in E is $x(t) = 0$

$\Rightarrow \quad M = \{0\} \Rightarrow$ AS

- LaSalle's thm is also useful in convergence to an eqlb. set

Example: (adaptive control)

- Suppose we like to stabilize the system

$$\dot{x} = \theta^* x + u$$ ← control input

by designing the control

- If θ^* is known, we can set $u = -(\theta^* + 1)x$ so that $\dot{x} = -x$ ⤳ stable

- Assume θ^* is unknown. Let $u = -(\hat{\theta} + 1)x$ where $\hat{\theta}$ is an estimate of θ^*

- We adapt $\hat{\theta}$ according to $\dot{\hat{\theta}} = x^2$ so that $\hat{\theta}$ becomes large when x is not converging to zero

- closed loop sys.

$$\dot{x} = (\theta^* - \hat{\theta} - 1)x$$

$$\dot{\hat{\theta}} = x^2$$

- Candidate Lyapunov func.

$$V(x, \hat{\theta}) = \frac{x^2}{2} + \frac{(\hat{\theta} - \theta^*)^2}{2}$$

- Therefore,

$$\dot{V}(x, \hat{\theta}) = [x, \ \hat{\theta} - \theta^*] \begin{bmatrix} (\theta^* - \hat{\theta} - 1)\, x \\ x^2 \end{bmatrix}$$

$$= (\theta^* - \hat{\theta} - 1)\, x^2 + (\hat{\theta} - \theta^*)\, x^2$$

$$= -x^2 \leq 0$$

- $$E = \{ (x, \hat{\theta}) \mid \dot{V}(x, \hat{\theta}) = 0 \} = \left\{ (x, \hat{\theta}) \;\middle|\; \begin{matrix} x = 0 \\ \hat{\theta} \in \mathbb{R} \end{matrix} \right\}$$

- By LaSalle, $(x(t), \hat{\theta}(t)) \to M \subset E \Rightarrow \boxed{x(t) \to 0}$

- The largest invariant set in E, is E itself

$$x(t) = 0 \Rightarrow \dot{\hat{\theta}}(t) = x(t)^2 = 0$$